PG–377 PGDMAT–11/ MMS–15

P.G. DIPLOMA IN MATHEMATICS EXAMINATION – JUNE 2019.

ALGEBRA

Time : 3 hours Maximum marks : 75

PART A — (5 × 5 = 25 marks)

Answer any FIVE questions.

1. Prove that a non-empty subset H of the group G is a subgroup of G if and only if

   (a) $a, b \in H$ implies that $\dot{a}\dot{b} \in H$.

   (b) $a \in H$ implies that $a^{-1} \in H$.

2. If G is a finite group and $a \in G$, then prove that $O(a) / O(G)$.

3. If R is a ring, then prove that for all $a, b \in R$

   (a) $AO = OA = 0$.

   (b) $a(-b) = (-a)b = -(ab)$.

4. If R is a commutative ring with unit element and M is an ideal of R, then prove that M is a maximal ideal of R if $\frac{R}{M}$ is a field.

5. If V is the internal direct sum of $U_1, U_2,..,U_n$, then prove that V is isomorphic to the external direct sum of $U_1, U_2,..,U_n$ where $U_1, U_2,..,U_n$ are subspaces of the vector space V.

6. If V is a finite dimensional inner product space and W is a subspace of V, then prove that $W^{\perp})^{\perp} = W$.

7. If $f(x) \in F[x]$ is of degree $n \geq 1$, then prove that there is an extension E of F of degree atmost $n\,!$ in which $f(x)$ has n-roots.

8. If $T \in A(V)$ and $S \in A(V)$ is regular, then prove that STS^{-1} and T have the same minimal polynomial

PART B — (5 × 10 = 50 marks)

Answer any FIVE questions.

9. State and prove Lagrange's theorem.

10. If G is a finite group, then prove that $C_a = O(G)/O(N(a))$.

11. Prove that the ideal $A = (a_0)$ is a maximal ideal of the Euclidean ring *R*. If and only if a_o is a prime element of *R*.

12. If *V* is a finite dimensional vector space and if *W* is a subspace of *V*, then prove that *W* is finite dimensional, $\dim W \leq \dim V$ and $\dim V / W = \dim V - \dim W$.

13. If *V* and *W* are of dimensions *m* and *n* respectively over *F*, then prove that $Hom(V, W)$ is of dimension *mn* over *F*.

14. If *F* is of characteristic '*O*' and if a, b are algebraic over *F*, then prove that there exists an element $C \in F(a, b)$ such that $F(a, b) = F(c)$.

15. (a) Prove that element $\lambda \in F$ is a characteristic root of $T \in A(V)$ if and only if there exists some $v \neq 0$ in *V*, $vT = \lambda v$.

    (b) If $\lambda \in F$ is a characteristic root of $T \in A(V)$, then prove that λ is a root of the minimal polynomial of *T*.

16. Prove that there exists a subspace *W* of *V* invariant under *T* such that $V = V_1 \oplus W$.

PG–378	**MMS-16 / PGDMAT–12**

M.Sc. (Mathematics)/P.G. DIPLOMA IN MATHEMATICS EXAMINATION – JUNE, 2019.

First Year

REAL ANALYSIS

Time : 3 hours Maximum marks : 75

PART A — (5 × 5 = 25 marks)

Answer any FIVE questions.

1. State and prove the Archimedean property of R.

2. Prove that a set E is open if and only if its compliment is closed.

3. Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then prove that $f(X)$ is compact.

4. Let f be monotonic on (a, b). Then prove that the set of points of (a, b) at which f is discontinuous is atmost countable.

5. Let f be defined on $[a, b]$. If f has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists then prove that $f'(x) = 0$.

6. If p^* is a refinement of a partition p, then prove that $L(p, f, \alpha) \leq L(p^*, f, \alpha)$.

7. State and prove the fundamental theorem of calculus.

8. If $x > 0$ and $y > 0$, then prove that

$$\int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = p(x, y).$$

PART B — (5 × 10 = 50 marks)

Answer any FIVE questions.

9. (a) Prove that closed subsets of compact sets are compact.

(b) Prove that the series $\sum a_n$

(i) Converges if $\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(ii) Diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

10. Suppose $\{s_n\},\{t_n\}$ are complex sequences and $\lim_{n\to\infty} s_n = S$, $\lim_{n\to\infty} t_n = t$ Then prove that

(a) $\lim_{n\to\infty}(s_n + t_n) = s + t$

(b) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty}(c + s_n) = c + s$ for any number c.

(c) $\lim_{n\to\infty} s_n t_n = st$.

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ provided $s_n \neq 0$ $(n = 1, 2, 3, ...)$ and $s \neq 0$.

11. (a) Define a continuous map on a metric space X.

(b) Let f be a continuous mapping of a compact metric space X into a metric space Y. Then prove that if is uniformly continuous on X.

12. State and prove the Taylor's theorem.

13. Assume α increases monotonically and $\alpha' \in R$ on $[a, b]$. Let f be a bounded real function on $[a, b]$. Then prove that $f \in R(\alpha)$ if and only if $f\alpha' \in R$. In that case $\int_a^b f\,dx = \int_a^b f(x)\alpha'(x)dx$.

14. State and prove the Stone-Weierstrass theorem.

15. Suppose $\sum c_n$ converges, put

$f(x) = \sum_{n=0}^{\infty} c_n x^n \ (-1 < x < 1)$ then prove that $\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n$.

16. (a) Define a contraction map.

(b) If X is a complete metric space and ϕ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\phi(x) = x$.

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PG-441	PGDMAT-13

P.G. DIPLOMA EXAMINATION – JUNE 2019.

TOPOLOGY AND FUNCTIONAL ANALYSIS

Time : 3 hours Maximum marks : 75

PART A — (5 × 5 = 25 marks)

Answer any FIVE questions.

1. If $\mathcal{B}$ is a basis for the topology of x and $\mathcal{C}$ is a basis for the topology of Y, then prove that the collection $\mathcal{D} = \{B \times C / B \in \mathcal{B} \, and \; C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

2. Prove that the image of a connected space under a continuous map is connected.

3. Show by an example that the product of two Lindelof spaces used not be lindelof.

4. Suppose that X has a countable basis. Then prove that every open covering of X contains a countable sub collection covering X.

5. Prove that the space lp^n of all n-tuples $x = (x_1 \ x_2 \ldots\ldots x_n)$ of scalars with the norm defined by $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ is a banach space.

6. State and prove the uniform boundedness theorem.

7. If T is an operator on a Hilbert space H, then prove that T is normal $\Leftrightarrow$ its real and imaginary parts commute.

8. State and prove the Schwarz inequality a Hilbert space H.

PART B — (5 × 10 = 50 marks)

Answer any FIVE questions.

9. Let X be a topological space, then show that the following conditions hold:

   (a) ϕ and X are closed

   (b) arbitrary intersections of closed sets is closed.

   (c) finite unions of closed sets is closed.

10. Show that finite Cartesian product of connected spaces is connected.

11. Prove that a subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded in the Euclidean metric 'd' or the square metric 'p'.

12. Let $\{U_1^\phi, U_2 ....U_n\}$ be a finite open covering of the normal space X. Then prove that there exists a partition of unity dominated by $\{Ui\}$.

13. Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x+M$ in the quotient space N/M is defined by $\| x+M \| = \inf\{\| x+m \| / \ m \in n\}$ then prove that N/M is a normed linear space. If N is a Banach space, then prove that N/M is a Banach space.

14. State and prove the closed graph theorem.

15. State and prove the Bessel's inequality in a Hilbert space H.

16. (a) Prove that an Operator T on a Hilbert space H is unitary if and only if it is an isometric isomorphism of H onto itself.

(b) Prove that a closed linear subspace M of H is invariant under an operator $T \Leftrightarrow M^{\perp}$ is invariant under T^*. (5+5)
